Journal of Statistical Physics, Vol. 104, Nos. 1/2, 2001

Long-Distance Transmission at Zero Dispersion: Exact Expressions for One-Time Statistical Properties

R. Blanco[1] **and L. Pesquera**[2]

Received August 23, 2000

We study the transmission of a signal through a dispersionless fiber with in-line amplifiers. The nonlinearities of the fiber and the noise generated at each amplifier are taken into account. Exact analytical expressions are obtained for the field averages. From these results we obtain the distributed amplification approximation, and study the conditions under which this approximation is valid.

KEY WORDS: Fiber optics; dispersion-free propagation; spontaneous-emission noise; statistical properties.

1. INTRODUCTION

In long (transoceanic) lightwave systems with optical amplifiers fiber nonlinearities are important, even for the low powers used in optical communications. In particular, Kerr nonlinearity and spontaneous emission noise of the in-line amplifiers produce large effects when they act together at the zero-dispersion point of a fiber. An analytical study of the transmission of a signal through a dispersionless fiber with in-line amplifiers has been undertaken by assuming that the amplifier spacing is much shorter than the nonlinear length of the fiber.[(1)] This allows to substitute the independent noises generated at each amplifier by a noise distributed along the fiber that is delta correlated in space. This distributed amplification model has been used to obtain the maximum propagation distance for the safe operation of systems operating at zero dispersion.

In this paper we present the exact analytical solution for the original discrete system with periodic optical amplification. The delta-correlated

[1] Departamento de Física Moderna, Universidad de Cantabria, 39005 Santander, Spain.
[2] Instituto de Física de Cantabria, CSIC-Universidad de Cantabria, 39005 Santander, Spain.

spontaneous-emission noise generated at each amplifier is adequately filtered, in a way that its variance remains finite. We consider ideal squared filters with a bandwidth much smaller than the bandwidth of the optical amplifiers. We denote B the bandwidth of the filtered noise, which means that its spectrum is flat for $|\omega| < \pi B$ and vanishes outside. Our exact results for the field averages are used to obtain the limits of validity of the distributed amplification model. The effect of the amplifier spacing on the propagation of the field is also analyzed.

The paper is organized as follows. In Section 2 the model for the propagation along the fiber is introduced. An expression for the signal at any amplifier is given in Section 3. In Section 4 the exact analytical expressions for one-time statistical properties of the propagating field are derived. These results are used in Section 5 to give the asymptotic expression for the signal at long propagation distances. In Section 6 the effect of the amplifier spacing is studied and the limits of validity of the distributed amplification model are obtained.

2. THE SYSTEM

We are considering a communication system consisting of a fiber working at its zero-dispersion point. In order to compensate for the losses in the fiber, equally separated in-line amplifiers are included in the system. Let l be the distance between the amplifiers. We shall assume that the first amplifier is located the same distance l away from the source of the signal. Between amplifiers the evolution of the slowly varying amplitude of a single-mode linearly polarized field is governed by the following nonlinear equation[(2)]

$$\frac{\partial}{\partial z} U(z, t) = -\Gamma U(z, t) - j\delta\, |U(z, t)|^2\, U(z, t). \tag{2.1}$$

The first term on the right-hand side represents the fiber loss, and the last term represents the effect of the Kerr nonlinearity, j denoting the imaginary unit ($j = \sqrt{-1}$), and

$$\delta = \frac{2\pi}{\lambda}\, n_2\, \frac{1}{\mathscr{A}_{\text{eff}}}, \tag{2.2}$$

where n_2 is the nonlinear index, $\mathscr{A}_{\text{eff}}$ is the effective cross section of the fiber, and λ is the free-space wavelength at the carrier frequency ω_0.

At each amplifier two effects have to be considered, namely, the amplifying effect and the generation of noise coming from the spontaneous emission within the amplifier. As a consequence, and assuming that the spatial extension of the amplifiers is negligible as compared to l, the output of an amplifier is given in terms of its input, by

$$U_{\text{out}}(t) = G^{1/2} U_{\text{in}}(t) + s(t), \tag{2.3}$$

where

$$G = \exp(2\Gamma l) \tag{2.4}$$

is the amplifier gain, calculated so as to compensate exactly for the fiber loss, and $s(t)$ denotes a complex gaussian noise accounting for the filtered spontaneous emission noise of the amplifier. If we denote its real and imaginary parts by $W(t)$ and $Y(t)$ respectively, the first and second order moments at equal times are given by

$$\langle W(t) \rangle = \langle Y(t) \rangle = 0, \tag{2.5}$$

$$\langle W^2(t) \rangle = \langle Y^2(t) \rangle = \sigma^2 = \frac{Kl}{2}, \qquad \langle W(t)\, Y(t) \rangle = 0, \tag{2.6}$$

with

$$K = \hbar\omega_0 \frac{(G-1)}{l} B\theta_a, \tag{2.7}$$

where θ_a is the noise-enhancement factor that accounts for incomplete inversion and B is the bandwidth of the filtered noise (See refs. 1, 3, and 4). The quantity K is introduced for further comparison with the results obtained with the distributed amplification model[(1)] and represents essentially the amount of noise intensity per unit length.

Let us now introduce the following notation: $s_n = W_n + jY_{\underline{n}}$ denotes the noise generated at the nth amplifier, U_n the input signal and $\bar{U}_n$ the output signal at the same amplifier, N the number of amplifiers, and $f(x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ the joint density function of all the noises W_n and Y_n. The time dependence is not relevant since only one-time statistical properties will be calculated. We then have the following relations

$$\bar{U}_n = G^{1/2} U_n + s_n, \tag{2.8}$$

$$\langle W_n \rangle = \langle Y_n \rangle = 0, \tag{2.9}$$

$$\langle W_n W_{n'} \rangle = \langle Y_n Y_{n'} \rangle = \sigma^2 \delta_{nn'}, \qquad \langle W_n Y_{n'} \rangle = 0, \tag{2.10}$$

$$f(x_1, y_1, x_2, y_2, \ldots, x_N, y_N) = \frac{1}{(\sigma\sqrt{2\pi})^{2N}} \exp\left[-\frac{\sum_{i=1}^{N} (x_i^2 + y_i^2)}{2\sigma^2} \right]. \tag{2.11}$$

The signal between consecutive amplifiers can be obtained by solving the differential equation (2.1), and is given by

$$U(z) = \bar{U}_n e^{-\Gamma(z-nl)} \exp\left[-j\delta l \, |\bar{U}_n|^2 \left(\frac{1-e^{-2\Gamma(z-nl)}}{2\Gamma l} \right) \right], \tag{2.12}$$

if $nl < z < (n+1)\,l$, which leads to

$$U_{n+1} = \bar{U}_n e^{-\Gamma l - j\delta l r \, |\bar{U}_n|^2}, \tag{2.13}$$

where

$$r = \frac{1-e^{-2\Gamma l}}{2\Gamma l}. \tag{2.14}$$

3. GENERAL SOLUTION

Our aim in this section is to obtain the expression of the signal at any amplifier of the array in terms of the signal at the source and the parameters of the system. The general relations of the former section enable us to obtain the following

Theorem 3.1. Let U_0 be the signal at the source, $\bar{s}_0 = 0$, and $\bar{s}_n$, with $n = 1, 2, \ldots, N$, N gaussian independent noises statistically equivalent to s_n. Then, the signal U_n just before the nth amplifier is given by

$$U_n = e^{-\Gamma l} \left(U_0 + \sum_{i=0}^{n-1} \bar{s}_i \right) \exp\left\{ -j\delta r l \left[\sum_{k=1}^{n} \left| U_0 + \sum_{i=0}^{k-1} \bar{s}_i \right|^2 \right] \right\}, \tag{3.1}$$

Proof. We make the proof in three steps.

Step 1. First we can obtain from Eqs. (2.8) and (2.13) the following iteration relation

$$U_{n+1} = (U_n + s_n e^{-\Gamma l}) \exp(-j\delta r l \, |U_n e^{\Gamma l} + s_n|^2). \tag{3.2}$$

We now introduce the following quantities for $n \geqslant 1$:

$$\varepsilon_n := \exp\left[j\delta r l \sum_{k=0}^{n-1} |\bar{U}_k|^2 \right], \tag{3.3}$$

and

$$V_n := U_n \varepsilon_n e^{\Gamma l}. \tag{3.4}$$

For the sake of completeness we also define $V_0 = U_0$ and $\varepsilon_0 = 0$.

Lemma 3.1. $V_n = U_0 + \sum_{i=0}^{n-1} s_n \varepsilon_n$.

Proof. From (3.3) we get $\varepsilon_{n+1} = \varepsilon_n \exp(j\delta rl\, |\bar{U}_n|^2)$, whence, with the aid of Eqs. (3.2) and (3.4), it results

$$V_{n+1} = (e^{\Gamma l} U_n \varepsilon_n + s_n \varepsilon_n) \exp(j\delta rl\, |\bar{U}_n|^2) \exp(-j\delta rl\, |U_n e^{\Gamma l} + s_n|^2). \tag{3.5}$$

Now Eq. (2.8) and the definition of V_n give

$$V_{n+1} = V_n + s_n \varepsilon_n = \varepsilon_n \bar{U}_n, \tag{3.6}$$

whose solution completes the proof of the lemma. ▮

Step 2. Let us define $\bar{s}_n = s_n \varepsilon_n$. The statistical properties of these quantities are established in the following

Lemma 3.2. The $\bar{s}_n$ are gaussian noises statistically identical to the s_n .

Proof. From Lemma 3.1 we can write ε_n in terms of the noises s_n and the signal at the source U_0, which is noise-independent. From Eq. (2.8), the definitions of V_n and s_n and the Eq. (3.6), we have $|\bar{U}_k| = |V_{k+1}|$. As a consequence we get

$$\bar{s}_n = s_n \exp\left[j\delta rl \sum_{k=1}^{n} |V_k|^2 \right] = s_n \exp\left[j\delta rl \sum_{k=1}^{n} \left| U_0 + \sum_{i=0}^{n-1} \bar{s}_n \right|^2 \right]. \tag{3.7}$$

Let us denote, for $n \leqslant N$, $\mathbf{S}^{(n)} = (W_1, Y_1, W_2, Y_2,..., W_n, Y_n)$ and $\bar{\mathbf{S}}^{(n)} = (\bar{W}_1, \bar{Y}_1, \bar{W}_2, \bar{Y}_2,..., \bar{W}_n, \bar{Y}_n)$. And let g_n the vector function relating $\mathbf{S}^{(n)}$ and $\bar{\mathbf{S}}^{(n)}$ as given by Eq. (3.7): $\mathbf{S}^{(n)} = g_n(\bar{\mathbf{S}}^{(n)})$. We also know the density function of $\mathbf{S}^{(N)}$, given by Eq. (2.11), $f_{\mathbf{S}^{(N)}} = f$, and we want to calculate $f_{\bar{\mathbf{S}}^{(N)}}$. To do this we first prove that g_n is invertible and with unit Jacobian. The existence of g_n^{-1} consists in the possibility of expressing $\bar{\mathbf{S}}^{(n)}$ in terms of $\mathbf{S}^{(n)}$. And this is easily done by induction

$$s_1 = \bar{s}_1 \exp(-j\delta rl\,|U_0|^2) \Rightarrow \bar{s}_1 = s_1 \exp(j\delta rl\,|U_0|^2),$$

$$s_2 = \bar{s}_2 \exp[-j\delta rl(|U_0|^2 + |U_0 + \bar{s}_1|^2)] \Rightarrow$$

$$\bar{s}_2 = s_2 \exp[j\delta rl(|U_0|^2 + |U_0 + s_1 e^{j\delta rl\,|U_0|^2}|^2)],$$

and so on.

As concerns the jacobian, it can also be proved by induction on n. If we call J_n the jacobian matrix of $\mathbf{S}^{(n)}$ with respect to $\bar{\mathbf{S}}^{(n)}$, it is easy to prove that $\det(J_{n+1}) = \det(J_n)$. To see this note that s_n, when expressed in terms of the $\bar{s}$'s, does not depend on $\bar{s}_k$ with $k > n$, and its dependence on $\bar{s}_n$ is only linear. Due to this it is easy to see that

$$J_{n+1} = \left(\begin{array}{c|cc} J_n & \multicolumn{2}{c}{0} \\ \hline B & \cos\psi_n & \sin\psi_n \\ & -\sin\psi_n & \cos\psi_n \end{array}\right) \tag{3.8}$$

where 0 is the null $2n \times 2$ matrix, B another $2 \times 2n$ irrelevant matrix and ψ_n depends on $\bar{s}_k$ with $k = 1, 2, \ldots, n-1$. Finally, $J_1 = 1$.

Now we can write the relation between the density functions of $\mathbf{S}$ and $\bar{\mathbf{S}}$

$$f_{\bar{\mathbf{S}}^{(N)}}(\bar{z}) = f_{\mathbf{S}^{(N)}}(g_N(\bar{z})), \tag{3.9}$$

where $z = (x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ and similar for $\bar{z}$. We can end the proof of Lemma 3.2 taken into account that the density function f only depends on the combination $x_i^2 + y_i^2$ and this expression is conserved by the function g_n. So, we finally get $f_{\bar{\mathbf{S}}^{(N)}} = f$. ▮

Step 3. It just remains to put together Eqs. (3.3), and (3.4), as well as the two lemmas so as to end the proof of the Theorem 3.1. ▮

To end this section we prove the following

Corollary 3.1. Let $\lambda_0 = 0$, and, for $i \leqslant N$, $\lambda_i = \rho_i + jI_i$, $\mathbf{R} \equiv (\rho_1, \rho_2, \ldots, \rho_N)$ and $\mathbf{I} \equiv (I_1, I_2, \ldots, I_N)$ being independent equally distributed real random vectors with zero mean and covariance matrix, $\mathbf{C}^{(N)}$, given by $C_{ij}^{(N)} = \sigma^2 \min\{i, j\}$. Then

(a) the signal U_n just before the nth amplifier is given by

$$U_n = e^{-\Gamma l}(U_0 + \lambda_{n-1}) \exp\left\{-j\delta rl \sum_{k=0}^{n-1} |U_0 + \lambda_k|^2\right\}, \tag{3.10}$$

(b) The signal $\bar{U}_n$ just after the nth amplifier is given by

$$\bar{U}_n = (U_0 + \lambda_n) \exp\left\{ -j\delta r l \sum_{k=0}^{n-1} |U_0 + \lambda_k|^2 \right\}. \tag{3.11}$$

Proof. Just as we did with s_n, we introduce the real and the imaginary parts of $\bar{s}_n$, $\bar{W}_n$ and $\bar{Y}_n$. Now we define, for $i = 1, 2, \ldots, N$,

$$\rho_i = \sum_{j=1}^{i} \bar{W}_j, \qquad I_i = \sum_{j=1}^{i} \bar{Y}_j. \tag{3.12}$$

The statistical properties of these new random variables follow immediately from those of $\bar{W}_n$ and $\bar{Y}_n$. First, they are gaussian because of being linear combinations of gaussian noises. Then, their first two moments are easily obtained from those of $\bar{W}_n$ and $\bar{Y}_n$, Eqs. (2.9) and (2.10). With this it is straightforward to get Eq. (3.11) from Eq. (3.1) in Theorem 3.1. Finally, Eq. (3.12) can be obtained with the aid of Eq. (2.13) and the expression for U_n. ▮

It will also be useful to write the joint density function of **R** and **I**. According to this corollary we have

$$f_\lambda(x_1, y_1, x_2, y_2, \ldots, x_N, y_N) = f_d(x_1, x_2, \ldots, x_N)\, f_d(y_1, y_2, \ldots, y_N), \tag{3.13}$$

where

$$f_d(x_1, x_2, \ldots, x_N) = \frac{1}{(\sigma\sqrt{2\pi})^{2N}} \exp\left[-\frac{1}{2\sigma^2} \sum_{ij} G_{ij}^{(N)} x_i x_j \right], \tag{3.14}$$

is the density function of **R** and **I** separately. In that expression the matrix $\mathbf{G}^{(N)}$ is related to the inverse of the covariance matrix by $\mathbf{G}^{(N)}/\sigma^2 = (\mathbf{C})^{-1}$ and its elements are

$$G_{ij}^{(N)} = \begin{cases} 2 & \text{if} \quad i = j \neq N \\ -1 & \text{if} \quad |i-j| = 1 \\ 1 & \text{if} \quad i = j = N \\ 0 & \text{otherwise.} \end{cases} \tag{3.15}$$

In order to simplify notation we shall use in the following $\mathbf{G}$ instead $\mathbf{G}^{(N)}$ whenever this is not misleading.

4. ONE-TIME STATISTICAL PROPERTIES OF U_n

The one-time statistical properties of the signal at the end of the transmission line are easily calculated taking advantage of the gaussian property of the noises. From the expression obtained for U_n in the previous section, it is straightforward to get

$$\left\langle U_{N+1}^{n} U_{N+1}^{*m} \right\rangle = e^{-(n+m)\Gamma l} \frac{\partial^{n+m}}{\partial s_1^n \, \partial s_2^m} M_{n-m}^{(N+1)}(s_1, s_2) \bigg|_{s_1 = s_2 = 0}, \tag{4.1}$$

where

$$M_n^{(N+1)}(s_1, s_2) = \left\langle \exp\left[s_1(U_0 + \lambda_N) + s_2(U_0^* + \lambda_N^*) - j\delta r l n \sum_{k=0}^{N} |U_0 + \lambda_k|^2 \right] \right\rangle. \tag{4.2}$$

The following theorem, which is the central result of this paper, gives an explicit expression of this function:

Theorem 4.1. Let $U_k(z)$ be the Chebyshev polynomials[5] defined by $U_n(\cos\theta) = \sin[(n+1)\theta]/\sin\theta$. Let also $D_k(z) = U_k(z/2)$, $b_n = \delta r n l^2 K$,

$$F_{N,1}(b) = \frac{1}{U_N(1 + jb/2) - U_{N-1}(1 + jb/2)}, \tag{4.3a}$$

and

$$F_{N,2}(b) = \frac{U_N(1 + jb/2)}{U_N(1 + jb/2) - U_{N-1}(1 + jb/2)}. \tag{4.3b}$$

Then

$$M_n^{(1)}(s_1, s_2) = \exp[s_1 U_0 + s_2 U_0^* - j\delta r l n \, |U_0|^2], \tag{4.4}$$

and, for $N > 0$,

$$\begin{aligned} M_n^{(N+1)}(s_1, s_2) = F_{N,1}(b_n) \exp[&(s_1 U_0 + s_2 U_0^*) \, F_{N,1}(b_n) \\ &+ (K s_1 s_2 - j\delta r n \, |U_0|^2) \, l F_{N,2}(b_n) - K l s_1 s_2]. \end{aligned} \tag{4.5}$$

Since nothing in the proof appears to be relevant for the rest of the paper, this proof is relegated to the Appendix.

The parameter b_n is related to the distance $z_2 = (2\delta r K)^{-1/2}$ over which the spontaneous-emission power alone gives a Kerr phase shift of the order of unity.[1] In fact $b_n = n(l/z_2)^2/2$, whence, in particular, $b_2^{-1/2}$ corresponds to the number of amplifiers necessary to achieve a distance z_2. This number is a function of the amplifier spacing l, which can be obtained from the definition of b_n in Theorem 4.1 and Eqs. (2.7) and (2.14). We get

$$b_n = 2n \frac{\delta \hbar \omega_0 B \theta_a}{\Gamma} \sinh^2(\Gamma l). \tag{4.6}$$

Due to the small value of the nonlinear parameter δ, b_n will take small values provided that the amplifier spacing is not too large. For example, assuming the values taken in ref. 1, ($B = 270$ GHz, $\theta_a = 2$, $\lambda = 1.53$ μm, $\delta = 3.75 \times 10^{-3}$ mW^{-1} km^{-1}, $\Gamma = 0.024$ km^{-1}) we obtain for $l = 60$ km a value $b_2 = 1.74 \times 10^{-4}$. Then b_2 will take small values for amplifier spacings not longer than 220 km. (The numerical values given above between brackets will be used throughout the rest of the paper.)

It is easy now to give a general expression for the moments of U_N. We simply have to put the expression of $M_n^{(N)}$, (4.5), into Eq. (4.1), and make use of the Leibniz relation,

$$\frac{d^m}{dx^m} [f(x)\, g(x)] = \sum_{k=0}^{m} \binom{m}{k} \frac{d^k f(x)}{dx^k} \frac{d^{m-k} g(x)}{dx^{m-k}}. \tag{4.7}$$

The result is, for $m \leqslant n$,

$$\begin{aligned} \langle U_{N+1}^n U_{N+1}^{*m} \rangle = {} & e^{-(n+m)\Gamma l} \exp[-j\delta r(n-m)\, |U_0|^2\, l F_{N,2}(\delta r(n-m)\, l^2 K)] \\ & \times \sum_{j=0}^{m} K^j l^j (F_{N,2}(\delta r(n-m)\, l^2 K) - 1)^j\, j! \binom{n}{j} \binom{m}{j} \\ & \times F_{N,1}^{n+m+1-2j}(\delta r(n-m)\, l^2 K)\, U_0^{n-j} U_0^{*m-j}. \end{aligned} \tag{4.8}$$

The lowest, more relevant, moments are

$$\langle U_{N+1} \rangle = e^{-\Gamma l} U_0 F_{N,1}^2(\delta r l^2 K) \exp[-j\delta r l\, |U_0|^2\, F_{N,2}(\delta r l^2 K)], \tag{4.9}$$

$$\langle |U_{N+1}|^2 \rangle = e^{-2\Gamma l} [|U_0|^2 + K l (N+1)], \tag{4.10}$$

$$\langle U_{N+1}^2 \rangle = e^{-2\Gamma l} U_0^2 F_{N,1}^3 (2\delta r l^2 K) \exp[-2j\delta r l\, |U_0|^2\, F_{N,2}(2\delta r l^2 K)]. \tag{4.11}$$

5. BEHAVIOR AT LARGE DISTANCE

In this section we study the asymptotic expression of the signal when the number of amplifiers is big enough. We will see that, after a sufficiently high number of amplifiers, the signal decreases exponentially. We indeed have the following

Theorem 5.1. Let $\zeta(b) = 1 + jb/2 + \sqrt{jb - b^2/4}$, where the square root is such that both its real and imaginary parts are positive. It is easy to see that with this choice $|\zeta(b)|$ is larger than one if b is positive. For N big enough to have

$$|\zeta(b_n)|^{-2N} \ll 1 \tag{5.1}$$

the functions $F_{N,1}$ and $F_{N,2}$ behave according to

$$\begin{aligned} |F_{N,1}(b_n)|^2 &\simeq C_1(b_n) \exp[-2N \ln(|\zeta(b_n)|)] \\ &= C_1(b_n) \exp[-z \sqrt{2\delta rnK}/C_2(b_n)], \end{aligned} \tag{5.2}$$

and

$$F_{N,2}(b_n) \simeq \frac{\zeta(b_n)}{\zeta(b_n) - 1}, \tag{5.3}$$

where $z = (N+1)\, l \simeq Nl$ and

$$C_1(b_n) = \left| \frac{\zeta(b_n) + 1}{\zeta(b_n)} \right|^2, \qquad \text{and} \qquad C_2(b_n) = \frac{\sqrt{2b_n}}{\ln(|\zeta(b_n)|^2)}. \tag{5.4}$$

Proof. Remember first the definition of the Chebyshev polynomials: If $x = \cos\theta$, then $U_N(x) = \sin((N+1)\,\theta)/\sin\theta$. Let us introduce now $\zeta = e^{i\theta}$. Then

$$x = \frac{\zeta + \zeta^{-1}}{2} \qquad \text{and} \qquad U_N(x) = \frac{\zeta^{N+1} - \zeta^{-(N+1)}}{\zeta - \zeta^{-1}}. \tag{5.5}$$

It is straightforward to see that, if we put $x = 1 + jb/2$, the corresponding ζ is just $\zeta(b)$. In the following ζ will denote in fact $\zeta(b_n)$. Now, the conditions of the theorem state that the negative power of ζ in the second of equations

(5.5) is negligible against the positive power. Then, we can make the following approximations

$$U_N \simeq \frac{\zeta^{N+1}}{\zeta-\zeta^{-1}}, \quad U_N - U_{N-1} \simeq \frac{\zeta^{N+1}-\zeta^N}{\zeta-\zeta^{-1}} = \left(\frac{\zeta-1}{\zeta-\zeta^{-1}}\right)\zeta^N = \frac{\zeta}{\zeta+1}\zeta^N, \tag{5.6}$$

whence it immediately follows (5.3). Finally, putting $|\zeta^N|^{-2} = \exp[-2N \ln|\zeta|]$, $N = z/l$, and expressing l in terms of b_n we obtain (5.2). ▮

We ask now how big N must be in order to fulfill the condition of the theorem, namely, $|\zeta|^{-2N} \ll 1$. To be more specific, we will look for the smallest integer, say N_c, such that $|\zeta|^{-2N_c} \leqslant 0.01$. Introducing $\phi(b_n) = \ln(10)/\ln(|\zeta|)$, we have $N_c = [\phi(b_n)] + 1$, where $[x]$ denotes the integer part of x. It is easy to realize that $\phi(b)$ is a decreasing function. Moreover, for very small b_n ($b_n \ll 1$) we can approximate $\phi(b_n) \simeq 2\ln 10/\sqrt{2b_n}$ whereas for large b_n, $\phi \simeq \ln 10/\ln(b_n)$. In particular, for $b_n \geqslant 10$ $N_c = 1$.

Taking into account that $b_2^{-1/2}$ gives the number of amplifiers that corresponds to the distance z_2, we obtain that for small b_n the required distance to fulfill the condition of the theorem, $z_c \equiv (N_c+1)\,l$, is of the order of $4z_2$. Furthermore, provided l is not too large, this is also the characteristic length associated with the decay of the exponential in (5.2). The reason is that C_2 is of the order of 1 except for too large b_n, while C_1 always takes values between 1 and 4. For the parameter values considered in the previous section we obtain for $l = 60$ km the distance $z_c = 20820$ km if we fix our attention on the first-order moment ($n = 1$).

In order to have an idea of how the moments behave at large distance, we must replace $F_{N,1}$ and $F_{N,2}$ in (4.5) or in (4.8) by their approximations, (5.2) and (5.3). The fact that $F_{n,2}$ appears in an exponent puts another limit on the validity of the approximation. This can be written as

$$\delta rl\,|U_0|^2 \left| F_{N,2}(b_n) - \frac{\zeta(b_n)}{\zeta(b_n)-1} \right| \simeq \frac{\delta\,|U_0|^2}{2\Gamma}(1-e^{-2\Gamma l}) \left| \frac{\zeta(b_n)+1}{\zeta(b_n)(\zeta(b_n)-1)} \right| |\zeta(b_n)|^{-2N} \ll 1, \tag{5.7}$$

where the other restriction, given by Eq. (5.1), has been used. This is a new condition only in the case that the expression ahead of $|\zeta(b_n)|^{-2N}$ is greater than one. Again this condition depends on the parameters of the system. But now it also depends on the power of the injected signal. Assuming again the values used in ref. 1, and taking for l, for example, the values 60, 100, and 200 km, we have that the maximum power for this condition to be less restrictive than (5.1) is, respectively, 0.065, 0.172, and 2.33 mW. Then

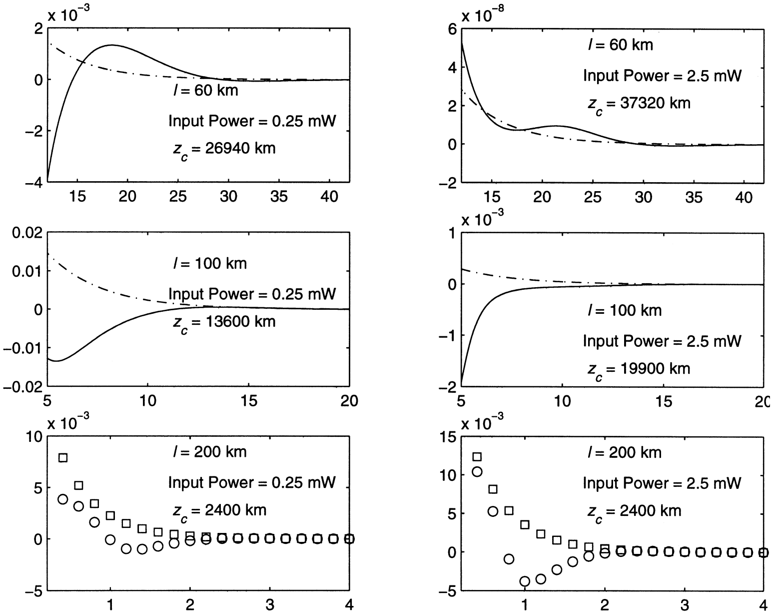

Fig. 1. Real part of the signal mean value at the position of the amplifiers for different values of the injected signal power and of the distance between amplifiers. In the cases $l = 60$ and 100 km a line plot is used since the amplifiers are too close at the scale of the figures. Exact expression: solid line or circle; large distance approximation: dash-dotted line or square. In each case the minimum distance for the approximation to be valid within an error of 1% is shown. Units are $\mathrm{mW}^{1/2}$ for the vertical axis and thousands of kilometers for the horizontal axis.

the behaviour at large distance depends also on the power when the amplifier spacing is small. This is due to the fact that the expression that multiplies $|\zeta(b_n)|^{-2N}$ in Eq. (5.7) behaves as $\exp(-\Gamma l)$ for small values of b_n. In Fig. 1 we present the exact values of the real part of the signal mean value, as compared to the large distance approximation, for different values of the signal power and of the distance between amplifiers. In each case the minimum distance to have an error smaller than 1%, z_c, is presented. This value is independent of the power only when the amplifier spacing is long enough.

6. COMPARISON WITH THE DISTRIBUTED AMPLIFICATION APPROXIMATION

Analytical results for the field averages have been obtained by Mecozzi[1] in the framework of the distributed amplification model.[1] In principle, this approximation requires the distance between amplifiers, l, to be much smaller than both the nonlinear length, $L_0 = 1/(\delta\,|U|^2)$, and the total length, z. By analyzing the steps leading from our exact expressions to Mecozzi's, more precise and rigorous conditions will turn out. Note first that the expression we have obtained for $M_n^{(N+1)}$, (4.5), and the one obtained by Mecozzi, Eq. (4.8) of ref. 1, own essentially the same structure, the difference lying in the specific functions $F_{N,1}$ and $F_{N,2}$. In order to compare our results with those obtained by Mecozzi,[1] we fix our attention on a given point of the transmission line. Let z be the given point, that we will assume to be at the entrance of the $N+1$ amplifier, that is $l(N+1) = z$. We indeed assume that a detector, and not an amplifier, is located at this position.

6.1. Obtaining the Approximation

We could be tempted to make the limit $l \to 0$ with $N \to \infty$ (so that $Nl = \text{const}$) in our exact expressions. However, in Mecozzi results we find some l-dependent quantities, namely, r and K. The reason is that, for realistic values of Γ and l, the product Γl is not small. We then have to consider l small but not strictly zero. In particular, we are not allowed to make the l-small approximation in the expressions of r and K.

Now, since the functions $F_{N,1}$ and $F_{N,2}$ depend on l through the parameter b_n, and the limit $l \to 0$ implies $b_n \to 0$, we are led to assume, as the first condition for obtaining the Mecozzi approximation, that b_n must be much smaller than one. (Note that the higher the moment order, the more restrictive this condition is. So, we will consider only low-order moments.) We recall that this parameter is given by $b_n = n(l/z_2)^2/2$. Then very small values of b_n are obtained when a large number of amplifiers are necessary to get a noise-induced Kerr phase shift of the order of unity. Then, remembering the definition of $\zeta(b)$ (see Theorem 5.1), for small b_n, ζ is close to 1, which allows to express ζ by means of a Taylor expansion. Due to the relation between x and ζ, see (5.5), one can see that the parameter of the expansion is $\sqrt{b_n}$, giving

$$\zeta = 1 + \sqrt{jb_n} + O(b_n). \tag{6.1}$$

Recall that in this expression we take the root with positive both real and imaginary parts, which corresponds to the case in which the modulus of ζ is larger than one. Now, the denominator of (5.5) can be approximated by

$$\zeta - \zeta^{-1} \simeq 2\sqrt{jb_n}\,. \tag{6.2}$$

This approximation cannot be made so naively in the numerator of (5.5) since N is large. The correct procedure is

$$\zeta^{N+1} \simeq (1+\sqrt{jb_n})^{N+1} = [(1+\sqrt{jb_n})^{1/\sqrt{jb_n}}]^{(N+1)\sqrt{jb_n}} \simeq e^{\sqrt{jb_n}(N+1)}. \tag{6.3}$$

With this we easily arrive at the following approximation for the Chebyshev polynomials

$$U_N(1+jb_n/2) \simeq \frac{\sinh(\sqrt{jb_n}(N+1))}{\sqrt{jb_n}}. \tag{6.4}$$

Now we put this in Eq. (4.3a) and make a Taylor expansion in powers of $\sqrt{b_n}$, taking into account that $\sqrt{b_n} \ll N\sqrt{b_n}$ but $N\sqrt{b_n}$ need not be small. Finally, using $(N+1)\sqrt{b_n} = (n\delta rK)^{1/2}\, z$, we obtain

$$F_{N,1}(n\delta rl^2K) \simeq \frac{1}{\cosh[(jn\delta rK)^{1/2}\, z]}, \tag{6.5}$$

which gives the Mecozzi expression for $F_{N,1}$.

For the other function, (4.3b), note that $F_{N,2}(b) = U_N(1+jb/2)\, F_{N,1}(b)$. Then, using (6.4), (6.5) and the definition of b_n we get

$$lF_{N,2}(n\delta rl^2K) \simeq \frac{\tanh[(jn\delta rK)^{1/2}\, z]}{(jn\delta rK)^{1/2}}. \tag{6.6}$$

which gives the Mecozzi expression for $F_{N,2}$.

6.2. Validity Conditions of the Approximation

Here we will analyze with more detail under which precise conditions the various approximations undertaken above are valid. First of all, b_n has to be small enough for the series expansion of ζ, (6.1), to be valid. For instance, if b_n is of the order of 10^{-4} the error in ζ would be of about 1%. We can write then

$$b_n \equiv n\delta rKl^2 \ll 1 \Rightarrow l \ll (n\delta rK)^{-1/2}, \tag{6.7}$$

and, if we are interested in low-order moments, the condition for the smallness of l results

$$l \ll (\delta rK)^{-1/2}, \tag{6.8}$$

which is of the order of the z_2 of ref. 1. Curiously enough, this condition is quite different to the one given by Mecozzi, $l \ll L_0 \equiv 1/(\delta\,|U|^2)$, for the validity of his approximation, in the sense that now it is the spontaneous-emission noise power, instead of the signal power, what puts a limit on the value of l. As we will see later, a variant of the Mecozzi condition has also to be fulfilled. Note, however, that if one wants to test the condition (6.8), one has to calculate r and K, which are l-dependent. This means that the true condition for l is obtained from (6.8) after expressing everything in terms of the actual quantities of the problem. The result is (see (4.6))

$$\sinh(\Gamma l) \ll \left(\frac{\Gamma}{2\delta\hbar\omega_0 B\theta_a}\right)^{1/2}. \tag{6.9}$$

For example, the parameters of ref. 1 give, for b not greater than 0.01, a maximum l of about 155 km.

There are further conditions for the validity of Mecozzi approximation. The first one concerns the possible values of N versus b_n. These quantities must be such that $(1+\sqrt{jb_n})^N$ can be approximated by the exponential of Eq. (6.3). First we need b_n to be small enough so that we can approximate $(1+\sqrt{jb_n})^{1/\sqrt{jb_n}}$ by e. For instance, if b_n is of the order of 10^{-4} the error in this step would be less than about 0.5%. This restriction for l (or b_n) is the same as that obtained above. Nevertheless the quantity just mentioned is raised to the Nth power. And in principle the error increases with N. Then this puts an upper limit on the possible values of N and b_n: for a given value of b_n, N cannot be arbitrarily big, and also, if we want the approximation to be valid for a certain value of N, then b_n cannot be arbitrarily big. In order to find how these quantities are restricted we are going to compare the approximate expression of ζ^N (see Eq. (6.3)), not with its Taylor expansion, but with the exact expression. Curiously enough the third term in (6.3) is much closer to the first one than the second term. We first realize numerically that $|\zeta|$ is greater than $|\exp(\sqrt{jb_n})| \equiv \exp(\sqrt{b_n/2})$ for values of b_n up to about 14. Consequently, since b_n must be small, the relative error is given by

$$\epsilon(b_n, N) = 1 - \left(\frac{e^{\sqrt{b_n/2}}}{|\zeta|}\right)^N. \tag{6.10}$$

The behavior of this error versus b_n is presented in Fig. 2 for various values of N. More interestingly, a plot of ϵ versus N is shown in Fig. 3 for different values of b_n. If we now define $N_m(b_n, \epsilon)$ as the largest possible value

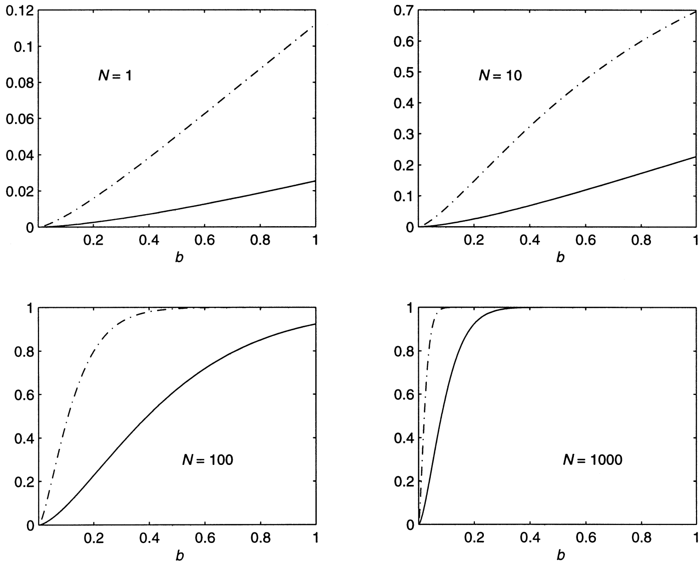

Fig. 2. Relative error made when the exact expression of ζ^N is replaced by $\exp[\sqrt{jb_n}(N+1)]$ (solid line) and by $(1+\sqrt{jb_n})^{N+1}$ (dash-dotted line). The error is plotted against b_n for various values of N.

of N which, for a given value of b_n, gives a relative error not greater than ϵ, since the error increases with N, it is easy to see that N_m is the integer part of $\Phi(b_n, \epsilon)$, given by

$$\Phi(b_n, \epsilon) = \frac{\ln(1-\epsilon)}{\sqrt{b_n/2} - \ln|\zeta|}. \tag{6.11}$$

Since b_n must be small, a Taylor expansion of ζ in powers of b_n gives

$$\Phi(b_n, \epsilon) \simeq \frac{34\epsilon}{b_n^{3/2}}. \tag{6.12}$$

This allows an easy calculation of N_m. For instance, for very small values of b_n of the order of 10^{-4}, if we admit an error of about 1%, the maximum number of amplifiers can be of the order of 340,000. Even an error of 0.0001 allows about 3,000 amplifiers. This value of b_n corresponds to an amplifier spacing of about 60 km. In practice, fewer amplifiers are necessary and so we can say that this does not put a severe limitation on the applicability of Mecozzi approximation when b_n is very small. However, this parameter increases in an exponential way with l (see (4.6)). If we

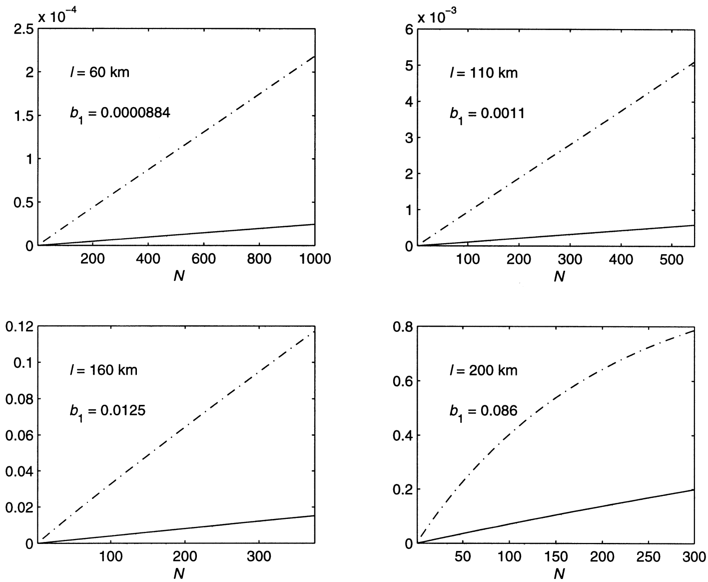

Fig. 3. The same as Fig. 2 but plotted against N for different values of b_n. We have shown the corresponding amplifier spacing for $n = 1$ and assumed the data of ref. 1, as said in the text. In every case the maximum distance equals 60,000 Km.

consider a value $l = 200$ km, we get $N_m \simeq 13$ for an error of about 1%, that corresponds to a distance $z_m \simeq 2600$ km. Then we can conclude that the distributed amplification approximation cannot be used to analyze trans-oceanic systems when the amplifier spacing is large. Note also that there is not a lower limit for the admissible values of N.

Up to now we have been studying the conditions under which the functions $F_{N,1}$ and $F_{N,2}$ can be approximated by the Mecozzi expressions. We now need these approximate expressions not to modify appreciably the values of the exact signal moments when they are used instead of the exact functions in (4.8). In particular the exponent of the exponential should not be very different. More specifically, we should impose

$$\delta r l \, |U_0|^2 \left| F_{N,2}(b_n) - \frac{\tanh(\sqrt{jb_n}(N+1))}{\sqrt{jb_n}} \right| \ll 1. \tag{6.13}$$

If we want this condition to be fulfilled for as large N as possible, the left hand side of this condition can be estimated by using its approximate expression in the limit of large N. Taking into account the results of the previous section, Eqs. (5.3) and (5.4), and the properties of the hyperbolic tangent, we have, for large N,

$$\left| F_{N,2}(b_n) - \frac{\tanh(\sqrt{jb_n}(N+1))}{\sqrt{jb_n}} \right| \simeq \left| \frac{\zeta}{\zeta-1} - \frac{1}{\sqrt{jb_n}} \right|, \tag{6.14}$$

and it is easy to see that this last expression takes values between 0.5 (for b_n small) and 1 (for b_n large). To have a better idea as to how the left hand side of this expression behaves, a plot of it versus N is presented in Fig. 4 for different values of b. For example, in the case of ref. 1 with $l = 60$ km, that expression equals 0.4 at 6000 km. We see that the maximum error made when $F_{N,2}$ is replaced by the Mecozzi expression lies between 0.5 and 1. Consequently, we obtain a new condition for the validity of the distributed amplification approximation, namely,

$$\delta r l \, |U_0|^2 \ll 1. \tag{6.15}$$

This is the only condition involving the intensity of the signal to be transmitted, and corresponds to the one referred to by Mecozzi, but with an effective nonlinear parameter δr, that takes into account the losses in

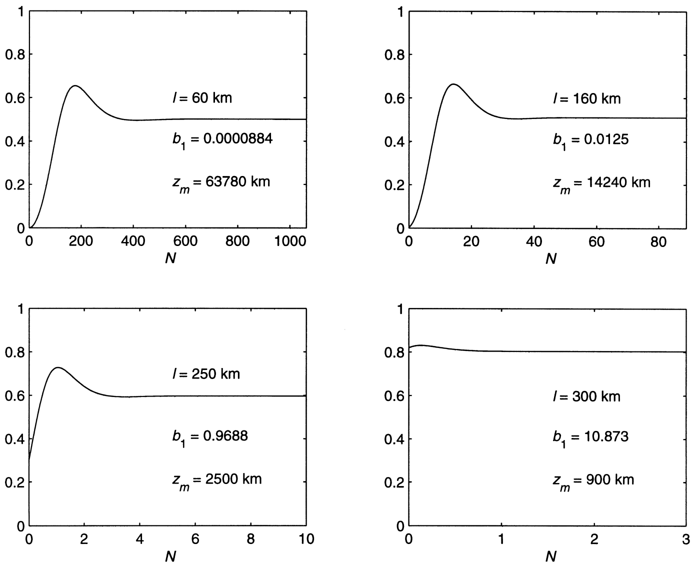

Fig. 4. Absolute error made in the substitution of $F_{N,2}$ by the Mecozzi expression, plotted against N for different values of b_n. Again the amplifier spacing is shown, as well as the maximum distance, z_m, for each plot.

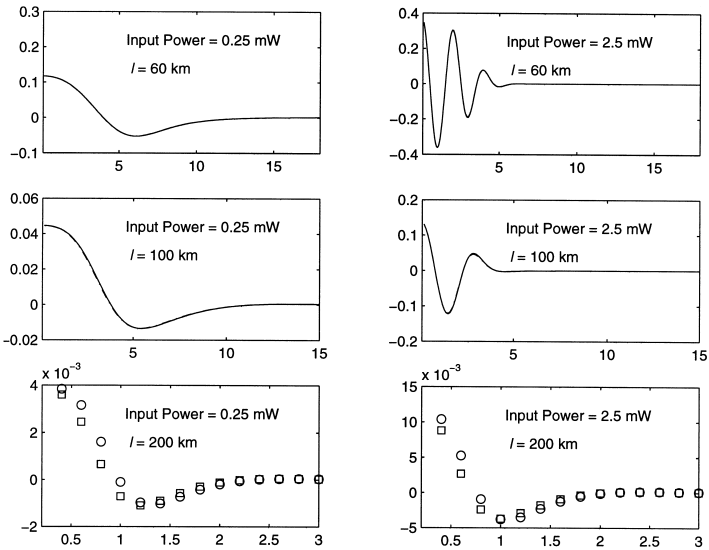

Fig. 5. Real part of the signal mean value for different values of the injected signal power and of the distance between amplifiers. For $l = 60$ and 100 km (again line plots; see caption of Fig. 1) there is no appreciable difference. For $l = 200$ km the error is considerable. Exact expression: solid line; Mecozzi approximation: dash-dotted line. Units are like in Fig 1.

the fiber, instead of δ. Since r depends on l that condition should rather be written as

$$(1 - \exp(-2\Gamma l)) \ll \frac{2\Gamma}{\delta\,|U_0|^2}. \tag{6.16}$$

We find in particular that, if the right hand side is much larger than one, we have no new restriction on the values of l. Note that this will happen whenever the signal intensity is small enough. However, this condition does not imply very low powers. For example, using the parameter values of Mecozzi's paper, the signal power has to be much smaller than about 13 mW, which is not too restrictive. For a power of 0.25 mW the error can be calculated to be of about 2%. In fact, for larger powers, an important error is introduced in these models because the effect of the filters upon the signal is not taken into account. This point will be discussed in a forthcoming paper. In Fig. 5 we present the behavior of the real part of the signal mean value for three values of l and two values of the injected signal.[3] We can realize that for $l = 200$ the error is considerable, even for

[3] Since we are considering the signal at the entrance of each amplifier, comparing our expressions with Mecozzi's requires to include in the latter the factor $\exp(-\Gamma l)$

low signal powers. This value of the amplifier spacing corresponds to $b_1 = 0.086$, which is not small enough.

7. CONCLUSIONS

We have studied the transmission of a signal through a dispersionless fiber with periodic optical amplification. The nonlinearities of the fiber and the noise generated at each amplifier have been taken into account in our analysis. We have obtained exact analytical expressions for one-time statistical properties of the propagating field. These results have been used to study signal propagation at long distances. We have also analyzed the limits of validity of the distributed amplification approximation.

It has been shown that the signal decreases exponentially at distances greater than z_2, that is the distance at which the noise-induced Kerr phase shift is of the order of unity. However, for small amplifier spacing the asymptotic behavior depends also on the power of the injected signal. In this case the distance at which the signal shows an exponential decay increases with the injected power.

We have shown that the distributed amplification approximation can be derived from our exact expressions when the amplifier spacing is much smaller than z_2. However, this approximation is only valid for propagation distances smaller than a maximum value. This maximum transmission distance decreases when the amplifier spacing increases. Finally, a condition involving the signal power has been derived for the validity of the distributed amplification approximation. This condition corresponds to the amplifier spacing being much smaller than the nonlinear length, but with an effective nonlinear parameter that takes into account the losses in the fiber. As a consequence the condition is satisfied irrespective of the amplifier spacing whenever the signal power is smaller than a certain value.

APPENDIX: PROOF OF THEOREM 4.1

The case $M_n^{(1)}$ is trivial. For the rest we shall make the proof in four steps.

Step 1. Taking into account that the real and the imaginary parts of the noises are statistically independent and equally distributed we can put

$$M_n^{(N+1)}(s_1, s_2) = e^{s_1 U_0 + s_2 U_0^*} \mathscr{A}_N(s_1 + s_2, \delta rln, \Re(U_0))\, \mathscr{A}_N(j(s_1 - s_2), \delta rln, \Im(U_0)) \tag{A.1}$$

where $\Re$ and $\Im$ denote respectively the real and the imaginary parts, and

$$\mathscr{A}_N(\alpha, a, u) = \left\langle \exp\left[\alpha\rho_N - ja \sum_{k=0}^{N} (u+\rho_k)^2 \right]\right\rangle. \tag{A.2}$$

Since the matrix $\mathbf{G}$ is real and symmetric, it is diagonalisable, its eigenvalues, μ_i, being real and its eigenvectors, $\mathbf{z}_i$, orthogonal. Let us denote $a_{ij} = (\mathbf{z}_j)_i$. We then have

Lemma A.1. Let the following auxiliary functions

$$g_1(b, N) = \sum_{r=1}^{N} \frac{a_{Nr}^2}{\mu_r + jb}, \quad g_2(b, N) = \sum_{r=1}^{N} \frac{a_{1r}^2}{\mu_r^2(\mu_r + jb)}, \quad g_3(b, N) = \sum_{r=1}^{N} \frac{a_{1r}a_{Nr}}{\mu_r(\mu_r + jb)}. \tag{A.3}$$

Then

$$\begin{aligned}
&\mathscr{A}_N(\alpha, a, u) \\
&\quad = e^{-ja(N+1)u^2}[\det(\mathbf{G} + jaKl\mathbf{I})]^{-1/2} \\
&\qquad \times \exp\left[\frac{Kl\alpha^2}{4} g_1(aKl, N) - Kla^2u^2 g_2(aKl, N) - jKl\alpha a u g_3(aKl, N) \right].
\end{aligned} \tag{A.4}$$

Proof. The function $\mathscr{A}_N$ can be calculated with the aid of the joint density function of the ρ's, given by (2.11),

$$\begin{aligned}
\mathscr{A}_N(\alpha, a, u) &= \int_{\mathbf{R}^N} dx_1\, dx_2 \cdots dx_N\, f_d(x_1, x_2, \ldots, x_N) \\
&\quad \times \exp\left[\alpha x_N - ja \sum_{k=0}^{N} (u + x_k)^2 \right] \\
&= e^{-ja(N+1)u^2} \frac{1}{(\sigma\sqrt{2\pi})^N} \int_{\mathbf{R}^N} dx_1\, dx_2 \cdots dx_N \\
&\quad \times \exp\left[\sum_{k=1}^{N} x_k C_k - ja \sum_{k=1}^{N} x_k^2 - \frac{1}{2\sigma^2} \sum_{ij} G_{ij} x_i x_j \right]
\end{aligned} \tag{A.5}$$

where

$$C_k = -2jau + \alpha\delta_{k,N}. \tag{A.6}$$

This is a gaussian integral that can be easily calculated by diagonalising the matrix **G**. Due to the above mentioned properties of **G** the corresponding transformation conserves the volume. Let ζ_i be the new coordinates. We have

$$x_i = \sum_j a_{ij}\zeta_j. \tag{A.7}$$

Then

$$\begin{aligned}
\mathscr{A}_N(\alpha, a, u) &= e^{-ja(N+1)u^2} \frac{1}{(\sigma\sqrt{2\pi})^N} \int_{\mathbf{R}^N} d\zeta_1\, d\zeta_2 \cdots d\zeta_N \\
&\quad \times \exp\left[\sum_{r=1}^{N} C_r \sum_{j=1}^{N} \zeta_j a_{rj} - ja \sum_{r=1}^{N} \zeta_r^2 - \frac{1}{2\sigma^2} \sum_{r=1}^{N} \mu_r \zeta_r^2 \right] \\
&= e^{-ja(N+1)u^2} \prod_{r=1}^{N} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} dy \exp\left[yA_r - jay^2 - \frac{1}{2\sigma^2}\mu_r y^2 \right] \\
&= e^{-ja(N+1)u^2} \prod_{r=1}^{N} \left(\frac{1}{\mu_r + jaKl} \right)^{1/2} \exp\left[\frac{KlA_r^2}{4(\mu_r + jaKl)} \right].
\end{aligned} \tag{A.8}$$

where

$$A_r = \sum_{j=1}^{N} C_j a_{jr}. \tag{A.9}$$

Now $\mu_r + jaKl$ are all nonvanishing eigenvalues of the matrix $\mathbf{G} + jaKl\mathbf{I}$, and then

$$\prod_{r=1}^{N} \left(\frac{1}{\mu_r + jaKl} \right)^{1/2} = [\det(\mathbf{G} + jaKl\mathbf{I})]^{-1/2}. \tag{A.10}$$

On the other hand, since $\mathbf{z}_k$ is eigenvector of **G** and μ_k its corresponding eigenvalue, and taking into account Corollary 3.1, we get

$$\frac{1}{\mu_r} a_{1r} \equiv \frac{1}{\mu_r} (\mathbf{z}_r)_1 = (\mathbf{G}^{-1}\mathbf{z}_r)_1 \equiv (\mathbf{C}\mathbf{z}_r)_1 = \sum_{j=1}^{N} C_{1j} (\mathbf{z}_r)_j = \sum_{j=1}^{N} a_{jr}, \tag{A.11}$$

and then we can express A_r as

$$A_r = \sum_{k=1}^{N} C_k a_{kr} = \sum_{k=1}^{N} (-2jau + \alpha\delta_{k,N})\, a_{kr}$$

$$= -2jau \sum_{k=1}^{N} a_{kr} + \alpha a_{Nr} = -2jau \frac{a_{1r}}{\mu_r} + \alpha a_{Nr}. \tag{A.12}$$

Putting this expression and (A.10) into Eq. (A.8) ends the proof of the lemma. ▌

In the following we shall calculate the three functions g.

Step 2.

Lemma A.2.

$$g_1(b, N) = [(\mathbf{G} + jb\mathbf{I})^{-1}]_{NN}, \tag{A.13a}$$

$$g_2(b, N) = \frac{N}{jb} + \frac{1}{b^2} - \frac{1}{b^2}[(\mathbf{G} + jb\mathbf{I})^{-1}]_{11}, \tag{A.13b}$$

$$g_3(b, N) = \frac{1}{jb} - \frac{1}{jb}[(\mathbf{G} + jb\mathbf{I})^{-1}]_{1N}. \tag{A.13c}$$

Proof. To prove the lemma we make use of the spectral theorem applied to $\mathbf{G}$, and take into account that the $\mathbf{z}$'s are eigenvectors in $\mathbf{R}^N$, and then real. Thus, for any f defined on the eigenvalues of $\mathbf{G}$

$$[f(\mathbf{G})]_{ij} = \sum_{r=1}^{N} f(\mu_r)(\mathbf{z}_r)_i\, (\mathbf{z}_r)_j = \sum_{r=1}^{N} f(\mu_r)\, a_{ir} a_{jr}. \tag{A.14}$$

Now, if we take $i = j = N$ and $f(\mathbf{G}) = (\mathbf{G} + jb)^{-1}$ we get

$$[(\mathbf{G} + jb\mathbf{I})^{-1}]_{NN} = \sum_{r=1}^{N} \frac{1}{\mu_r + jb}(a_{Nr})^2 = g_1(b, N), \tag{A.15}$$

which gives (A.13a). For the others we put, respectively, $f(\mathbf{G}) = \mathbf{G}^{-2}(\mathbf{G} + jb\mathbf{I})^{-1}$ with $i = j = 1$ and $f(\mathbf{G}) = \mathbf{G}^{-1}(\mathbf{G} + jb\mathbf{I})^{-1}$ with $i = 1$, $j = N$, getting

$$[\mathbf{G}^{-2}(\mathbf{G} + jb\mathbf{I})^{-1}]_{11} = \sum_{r=1}^{N} \frac{1}{\mu_r^2(\mu_r + jb)}(a_{1r})^2 = g_2(b, N), \tag{A.16}$$

and

$$[\mathbf{G}^{-1}(\mathbf{G}+jb\mathbf{I})^{-1}]_{1N} = \sum_{r=1}^{N} \frac{1}{\mu_r(\mu_r+jb)} a_{1r} a_{Nr} = g_3(b, N). \tag{A.17}$$

The following relations end the proof of the lemma

$$\mathbf{G}^{-1}(\mathbf{G}+jb\mathbf{I})^{-1} = \frac{1}{jb}[\mathbf{G}^{-1} - (\mathbf{G}+jb\mathbf{I})^{-1}], \tag{A.18a}$$

$$\mathbf{G}^{-2}(\mathbf{G}+jb\mathbf{I})^{-1} = \frac{1}{jb}\mathbf{G}^{-2} + \frac{1}{b^2}\mathbf{G}^{-1} - \frac{1}{b^2}(\mathbf{G}+jb\mathbf{I})^{-1}, \tag{A.18b}$$

$$(\mathbf{G}^{-1})_{11} = (\mathbf{G}^{-1})_{1N} = 1, \tag{A.18c}$$

$$(\mathbf{G}^{-2})_{11} = \sum_{i=1}^{N} (\mathbf{G}^{-1})_{1i} (\mathbf{G}^{-1})_{i1} = N. \quad \blacksquare \tag{A.18d}$$

Step 3. We need now to calculate the matrix elements of $(\mathbf{G}+jb\mathbf{I})^{-1}$. We shall prove the following

Lemma A.3. Let $U_k(z)$ be the Chebyshev polynomials (see Theorem 4.1) for $k \geqslant 0$, and $U_{-1}(z) \equiv 0$. Then

$$[\det(\mathbf{G}+jb\mathbf{I})]^{-1} = \frac{1}{U_N(1+jb/2) - U_{N-1}(1+jb/2)}, \tag{A.19a}$$

$$[(\mathbf{G}+jb\mathbf{I})^{-1}]_{11} = \frac{U_{N-1}(1+jb/2) - U_{N-2}(1+jb/2)}{U_N(1+jb/2) - U_{N-1}(1+jb/2)}, \tag{A.19b}$$

$$[(\mathbf{G}+jb\mathbf{I})^{-1}]_{1N} = \frac{1}{U_N(1+jb/2) - U_{N-1}(1+jb/2)}, \tag{A.19c}$$

$$[(\mathbf{G}+jb\mathbf{I})^{-1}]_{NN} = \frac{U_{N-1}(1+jb/2)}{U_N(1+jb/2) - U_{N-1}(1+jb/2)}. \tag{A.19d}$$

Proof. To calculate the elements of the inverse matrix of $\mathbf{G}+jb\mathbf{I}$ we introduce two families of matrices, $\mathbf{G}_n^{(1)}(\alpha)$ and $\mathbf{G}_n^{(2)}(\alpha)$, of dimensions $n \times n$ given by

$$(G_n^{(1)})_{ij}(\alpha) = \begin{cases} \alpha & \text{if} \quad i = j \neq n, \\ -1 & \text{if} \quad |i-j| = 1, \\ \alpha - 1 & \text{if} \quad i = j = n, \\ 0 & \text{otherwise.} \end{cases} \tag{A.20}$$

and $\mathbf{G}_n^{(2)}$ is the same as $\mathbf{G}_n^{(1)}$ except that $(G_n^{(2)})_{nn}(\alpha) = \alpha$ instead of $\alpha - 1$. Note that $\mathbf{G} + jb\mathbf{I} = \mathbf{G}_N^{(1)}(2 + jb)$. Let also denote $D_n(\alpha) = \det \mathbf{G}_n^{(2)}$. The following relation can be easily obtained from the definitions of the two matrices

$$\det \mathbf{G}_n^{(1)} = (\alpha - 1) \det \mathbf{G}_{n-1}^{(2)} - \det \mathbf{G}_{n-2}^{(2)}. \tag{A.21}$$

Moreover, the cofactor of the (1, 1)-element is $\det \mathbf{G}_{n-1}^{(1)}$, that of the (n, n)-element, $\det \mathbf{G}_{n-1}^{(2)}$, and that of the $(1, n)$-element is 1. Consequently we have

$$[(\mathbf{G}_n^{(1)})^{-1}]_{11} = \frac{\det \mathbf{G}_{n-1}^{(1)}}{\det \mathbf{G}_n^{(1)}} = \frac{(\alpha - 1) D_{n-2} - D_{n-3}}{(\alpha - 1) D_{n-1} - D_{n-2}}, \tag{A.22a}$$

$$[(\mathbf{G}_n^{(1)})^{-1}]_{1n} = \frac{1}{\det \mathbf{G}_n^{(1)}} = \frac{1}{(\alpha - 1) D_{n-1} - D_{n-2}}, \tag{A.22b}$$

$$[(\mathbf{G}_n^{(1)})^{-1}]_{nn} = \frac{\det \mathbf{G}_{n-1}^{(2)}}{\det \mathbf{G}_n^{(1)}} = \frac{D_{n-1}}{(\alpha - 1) D_{n-1} - D_{n-2}}. \tag{A.22c}$$

Note that, in principle, these expressions are valid only if $N > 3$, since D_n is defined for $n > 0$. We can, however, define $D_0 = 1$, $D_{-1} = 0$ and $D_{-2} = -1$ and expressions (A.22a)–(A.22c) become valid for $N > 0$.

Now, to calculate the determinant of $\mathbf{G}_n^{(2)}$, an expansion based on the first column, leads to the following recurrence relation

$$D_n = \alpha D_{n-1} - D_{n-2}, \tag{A.23}$$

for $n > 2$. And we can see that, with the former definitions, this expression is in fact valid for $n \geqslant 0$. This recurrence relation and the expressions for D_0 and D_1 prove that $D_n(2\alpha)$ are the Chebyshev polynomials (see ref. 5). This and the use of the recurrence relation in Eqs. (A.22a)–(A.22c) prove the lemma. ▌

Step 4. It only remains to put together all the results obtained previously. First we see that $\det(\mathbf{G} + jaKl\mathbf{I})^{-1}$, given by Eq. (A.19a), coincides with $F_{N,1}$, (4.3a). The functions g_1, g_2 and g_3 can be expressed in terms of the Chebyshev polynomials by using Eqs. (A.13a)–(A.13c) and (A.19b)–(A.19d). Then we put these expressions into Eq. (A.4), as well as $\det(\mathbf{G} + jaKl\mathbf{I})^{-1}$. Thus, Eq. (A.1) can be written now

$$\begin{aligned} M_n^{(N+1)}(s_1, s_2) = {} & \exp[s_1 U_0 + s_2 U_0^* - j\delta rln(N+1)\,|U_0|^2]\, F_{N,1}(\delta rnKl^2) \\ & \times \exp[Kls_1 s_2 g_1(\delta rnKl^2, N) - Kl^3\delta^2 r^2 n^2\,|U_0|^2\, g_2(\delta rnKl^2, N) \\ & - j\delta rnKl^2(s_1 U_0 + s_2 U_0^*)\, g_3(\delta rnKl^2, N)], \end{aligned} \tag{A.24}$$

which gives the desired result, Theorem 4.1, if we take into account that

$$1 - jbg_3(b, N) = [(\mathbf{G} + jb\mathbf{I})^{-1}]_{1N} = F_{N,1}(b), \tag{A.25}$$

$$g_1(b, N) = F_{N,2}(b) - 1, \tag{A.26}$$

and

$$(N+1) - jbg_2(b, N) = F_{N,2}(b), \tag{A.27}$$

where we have made use of Eqs. (4.3a), (4.3b), (A.13a)–(A.13c) and (A.19a)–(A.19d), as well as, for the last expression, the recurrence relation of the Chebyshev polynomials, $U_n(\alpha) = 2\alpha U_{n-1}(\alpha) - U_{n-2}(\alpha)$. ▮

ACKNOWLEDGMENTS

We acknowledge financial support by CICYT (Spain) Project Number TIC1999-0645-C05-01.

REFERENCES

1. A. Mecozzi, Long-distance transmission at zero dispersion: Combined effect of the Kerr nonlinearity and the noise of the in-line amplifiers, *J. Opt. Soc. Am. B* **11**:462–469 (1994).
2. G. P. Agrawal, *Nonlinear Fiber Optics* (Academic Press, 1989).
3. D. Marcuse, Single-channel operation in very long nonlinear fibers with optical amplifiers at zero dispersion, *J. Lightwave Technol.* **9**:356–361 (1991).
4. D. Marcuse, *Principles of Quantum Electronics* (Academic Press, 1980).
5. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, 1970).